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Accurate Semi-implicit Treatment of the Hall Effect in Magnetohydrodynamic Computations

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A semi-implicit operator is derived for the Hall term that, when implemented in the MHD equations, allows time steps comparable to those normally used in semi-implicit MHD computations. The operator is fourth order in the direction of the magnetic field and is most accurate when used in a split time step algorithm. The effects of the semi-implicit treatment on linear waves in a cylinder are shown both through a detailed analysis and numerical tests. The effectiveness of the algorithm is also demonstrated with a nonlinear computation of a tokamak tearing instability. © 1989 Academic Press, Inc.

1. INTRODUCTION

The modification of the magnetohydrodynamic (MHD) equations due to the Hall effect can be important in many applications. In present day reversed-field pinch (RFP) experiments the Hall term can become comparable to the velocity term in Ohm's law. The Hall term is also important in edge regions having sharp gradients and low density. In field-reversed configuration (FRC) experiments some theoretical explanations for the absence of tilt instabilities consider rotation to be a stabilizing effect [1]. In FRCs, however, a large contribution to rotation comes from the diamagnetic drift which appears only with the Hall effect. Explicit schemes for 3-dimensional time-dependent computations including the Hall effect are complicated by a very restrictive stability limit on the time step. The same time step limit also arises in quasineutral hybrid (particle ions, fluid electrons) computations [2, 3]. Implicit methods do not have such a time step restriction, but are difficult to implement due to the coupling of all the components in the induction equation.

In this paper we describe how to use a semi-implicit method to obtain an unconditionally stable method, with respect to the Hall term, without degrading the accuracy of the computation for time steps appropriate for resistive MHD computations. The method is similar to the semi-implicit MHD [4, 5] method, which is used to eliminate the Alfvén Courant–Friedrichs–Lewy (CFL) condition. In order to stabilize the Hall term, the semi-implicit operator acts on the magnetic field advance. The correct choice of operator is found to be essential in order to maintain accuracy.

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2. Semi-implicit Model

The compressible resistive MHD equations with the Hall term may be written

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left[\mathbf{v} \times \mathbf{B} - \frac{1}{\omega_{ci} \tau_A \rho} (\mathbf{J} \times \mathbf{B} - \nabla P_e) \right] - \frac{1}{S} \nabla \times \eta \mathbf{J}, \tag{1}$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \mathbf{J} \times \mathbf{B} - \nabla P, \tag{2}$$

$$\frac{\partial P}{\partial t} = -\mathbf{v} \cdot \nabla P - \gamma P \nabla \cdot \mathbf{v}, \tag{3}$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}), \tag{4}$$

$$\mathbf{J} = \nabla \times \mathbf{B}, \tag{5}$$

where $\mathbf{v}$ is the velocity, $\mathbf{B}$ is the magnetic field, $\mathbf{J}$ is the current density, P is the plasma pressure, P_e is the electron component of the plasma pressure, ρ is the density, and η is the resistivity. The equations have been written in dimensionless variables with $\mathbf{B} = \mathbf{B}'/B_0$, $\mathbf{J} = \mathbf{J}'(4\pi L/B_0 c)$, $x = x'/L$, $t = t'/\tau_A$, $\rho = \rho'/\rho_0$, $P = P'(4\pi/B_0^2)$, $\eta = \eta'/\eta_0$, and $S = \tau_R/\tau_A$. The primed variables are the physical quantities and the unprimed are in dimensionless units. The Alfvén transit time is defined by $\tau_A = L/V_A$, where V_A is the Alfvén speed, $V_A = B_0/\sqrt{4\pi\rho_0}$; ω_{ci} is the ion-cyclotron frequency, $\omega_{ci} = (qB_0/m_i c)$; τ_R is the resistive decay time, $\tau_R = (4\pi L^2/\eta_0 c^2)$; γ is the ratio of specific heats; B_0, ρ_0, η_0, and L are convenient normalizing values.

The significance of the Hall term is determined by the magnitude of $(\omega_{ci}\tau_A)^{-1}$. This term is large in regions with low densities or short scale lengths. For example, in the ZT-40M RFP [6] with a density of $2 \times 10^{13}\,\mathrm{cm}^{-3}$ and a minor radius of 25 cm, this factor is approximately 0.3 and is too large to neglect.

In order to examine the effect of the Hall term on the numerical stability of time-dependent computations, we consider the case in which $(\omega_{ci}\tau_A)^{-1}$ is very large and drop the velocity term from the induction equation, Eq. (1). We also neglect the electron pressure gradient. Then the induction equation decouples from the other MHD equations. To derive the form of the semi-implicit operator, we linearize Eq. (1) about a uniform density plasma in a uniform magnetic field, $\mathbf{B}_0$, to obtain

$$\frac{\partial \mathbf{B}_1}{\partial t} = -\frac{1}{\omega_{ci}\tau_A} \nabla \times (\mathbf{J}_1 \times \mathbf{B}_0). \tag{6}$$

Differentiating with respect to time and using $\nabla \cdot \mathbf{B} = 0$ gives (see the Appendix)

$$\frac{\partial^2 \mathbf{B}_1}{\partial t^2} = -\frac{1}{(\omega_{ci}\tau_A)^2} (\mathbf{B}_0 \cdot \nabla)^2 \nabla^2 \mathbf{B}_1. \tag{7}$$

This fourth-order operator guides the choice of the semi-implicit operator for eliminating numerical instabilities associated with the Hall term.

The procedure for developing a semi-implicit time advance is identical to that used in the semi-implicit MHD method, except that the semi-implicit treatment is on the magnetic field advance rather than the velocity advance [2, 3, 5]. The operator on the right side of Eq. (7) is used with a vector coefficient, to be determined from stability considerations, as an approximation to the effect of the Hall term in the second-order differential equation. This operator is then added to each side of the induction advance. Consider the case with no pressure, no resistivity, and advection neglected in the momentum equation. In order to eliminate the CFL constraint due to the Alfvén waves, the velocity equation is treated semi-implicitly and, for simplicity, the operator is chosen to be the Laplacian. Using a leapfrog time discretization with semi-implicit terms for both the Alfvén [7] and Hall terms, one may use either a single-step semi-implicit advance of the induction equation or one using operator splitting.

2.1. *Single-Step Method*

Applying the form of the operator in Eq. (7) with constant semi-implicit coefficients directly to the induction equation gives the leapfrog advance

$$\begin{aligned}
&\mathbf{B}^{n+1} + (\Delta t)^2 (\mathbf{C}_H \cdot \nabla)^2 \nabla^2 \mathbf{B}^{n+1} \\
&\quad = \mathbf{B}^n + (\Delta t)^2 (\mathbf{C}_H \cdot \nabla)^2 \nabla^2 \mathbf{B}^n \\
&\qquad + \Delta t \nabla \times \left(\mathbf{v}^{n+1/2} \times \mathbf{B}^{n+1/2} - \frac{1}{\omega_{ci} \tau_A \rho} \mathbf{J}^{n+1/2} \times \mathbf{B}^{n+1/2} \right), \qquad (8)
\end{aligned}$$

$$\begin{aligned}
&\mathbf{v}^{n+1} - (\Delta t)^2 C_A^2 \nabla^2 \mathbf{v}^{n+1} \\
&\quad = \mathbf{v}^n - (\Delta t)^2 C_A^2 \nabla^2 \mathbf{v}^n + (\Delta t/\rho) \mathbf{J}^{n+1/2} \times \mathbf{B}^{n+1/2}. \qquad (9)
\end{aligned}$$

C_A and $\mathbf{C}_H$ are the Alfvén and Hall semi-implicit parameters, respectively. The above algorithm is second-order accurate in time and has no dissipation. Dissipation may be added if it is desired. To analyze the stability of this algorithm, consider a linearlized problem in one dimension ($\partial/\partial x = 0$, $\partial/\partial y = 0$) with a uniform equilibrium magnetic field, $\mathbf{B}_0 = B_0 \mathbf{z}$ and a uniform density. Assuming that all quantities vary as $e^{i(\omega t + k_z z)}$, we obtain

$$2 \sin\left(\frac{\omega \Delta t}{2}\right) S_A v_x = \Delta t k_z B_x B_0, \qquad (10)$$

$$2 \sin\left(\frac{\omega \Delta t}{2}\right) S_A v_y = \Delta t k_z B_y B_0, \qquad (11)$$

$$2 \sin\left(\frac{\omega \Delta t}{2}\right) S_H B_x = \Delta t k_z \left(v_x + i \frac{k_z B_y}{\omega_{ci} \tau_A} \right) B_0, \qquad (12)$$

$$2 \sin\left(\frac{\omega \Delta t}{2}\right) S_H B_y = \Delta t k_z \left(v_y - i \frac{k_z B_x}{\omega_{ci} \tau_A} \right) B_0, \qquad (13)$$

where we have defined $S_A = 1 + (\Delta t)^2 C_A^2 k_z^2$ and $S_H = 1 + (\Delta t)^2 C_H^2 k_z^4$. $\mathbf{C}_H$ has been chosen so that $\mathbf{C}_H = C_H \mathbf{z}$. After some algebra the following dispersion relation is obtained,

$$\sin^2\left(\frac{\omega \Delta t}{2}\right) = \frac{1}{8}(2X + Y \pm \sqrt{4XY + Y^2}), \tag{14}$$

where $X = (\Delta t)^2 k_z^2 B_0^2 / S_A S_H$ and $Y = (\Delta t)^2 k_z^4 B_0^2 / (S_H \omega_{ci} \tau_A)^2$. For stability the right side of Eq. (14) must be less than or equal to one. In the limit $\omega_{ci}\tau_A \to \infty$ (i.e., no Hall term) we have $Y = 0$ and we can choose $C_H = 0$, for which Eq. (14) gives stability for $X < 4$. For unconditional stability, we require $(\Delta t)^2 k_z^2 B_0^2 / (1 + (\Delta t)^2 C_A^2 k_z^2) < 4$. This is satisfied for all k_z if $C_A > |B_0|/2$. This is just the usual condition for stability in the semi-implicit MHD method. The other limit, with $\omega_{ci}\tau_A \to 0$, so that $X \ll Y$ (i.e., the Hall term dominates), gives stability for $Y < 4$. For unconditional stability this requires that

$$\frac{(\Delta t)^2 k_z^4 B_0^2 (\omega_{ic}\tau_A)^{-2}}{1 + 2(\Delta t)^2 C_H^2 k_z^4 + (\Delta t)^4 C_H^4 k_z^8} < 4. \tag{15}$$

In order to satisfy this condition for all k_z it is necessary to have $C_H > (1/4)|B_0|/(\omega_{ci}\tau_A)$. These criteria on C_A and C_H are not sufficient for stability at intermediate values of $\omega_{ci}\tau_A$. Numerical solutions of the dispersion relation (14) show that numerical stability can be achieved at all values of $\omega_{ci}\tau_A$ if $C_A > |B_0|/\sqrt{2}$, and $C_H > (1/2\sqrt{2})|B_0|/(\omega_{ci}\tau_A)$. These enhanced values of the semi-implicit coefficients are required to stabilize modes with very large k_z; if these high k_z modes are not present in the calculation, correspondingly smaller values of C_A and C_H may be used.

Another possible choice for a Hall term semi-implicit operator is simply the Laplacian [8]. Equation (8) becomes

$$\mathbf{B}^{n+1} - \Delta t C_H \nabla^2 \mathbf{B}^{n+1}$$

$$= \mathbf{B}^n - \Delta t C_H \nabla^2 \mathbf{B}^n + \Delta t \nabla \times \left(\mathbf{v}^{n+1/2} \times \mathbf{B}^{n+1/2} - \frac{1}{\omega_{ci}\tau_A \rho} \mathbf{J}^{n+1/2} \times \mathbf{B}^{n+1/2} \right). \tag{16}$$

The semi-implicit terms make this algorithm only first-order accurate in time. It is still, however, possible to obtain unconditional stability with this algorithm. The dispersion relation for the 1-dimensional problem is identical to Eq. (14), except for the fact that now $S_H = 1 + \Delta t C_H k_z^2$. Unconditional stability in the limit of $X \to 0$, therefore, requires

$$\frac{(\Delta t)^2 k_z^4 B_0^2 (\omega_{ci}\tau_A)^{-2}}{1 + 2\Delta t C_H k_z^2 + (\Delta t)^2 C_H^2 k_z^4} < 4. \tag{17}$$

This condition is satisfied for all k_z if $C_H > \frac{1}{2}|B_0|/(\omega_{ci}\tau_A)$. The error associated with this first-order scheme can be too large to make it useful. The highly dispersive

character of algorithms using the Laplacian for the Hall term will be shown in the numerical tests.

2.2. *Split Methods*

The same semi-implicit operators described in Section 2.1 may be used in a two-stage advance of the induction equation. In the two-stage method the advance is split between the MHD part and the Hall term. The advance of the magnetic field in Eq. (8) is replaced by

$$\mathbf{B}^* = \mathbf{B}^n + \Delta t\, \nabla \times (\mathbf{v}^{n+1/2} \times \mathbf{B}^{n+1/2}), \tag{18}$$

$$\mathbf{B}^{n+1} + (\Delta t)^2 (\mathbf{C}_H \cdot \nabla)^2 \nabla^2 \mathbf{B}^{n+1}$$

$$= \mathbf{B}^* + (\Delta t)^2 (\mathbf{C}_H \cdot \nabla)^2 \nabla^2 \mathbf{B}^* - \frac{\Delta t}{\omega_{ci} \tau_A \rho} \nabla \times (\mathbf{J}^{n+1/2} \times \mathbf{B}^{n+1/2}). \tag{19}$$

An important feature of this scheme is that the semi-implicit operator for the Hall term only alters the Hall contribution. It has no direct effect on the original MHD equations. Therefore, in the limit of the Hall term going to zero or the Hall semi-implicit terms becoming very large (i.e., large Δt), the solution to the resistive MHD advance without the Hall term is recovered. The dispersion relation for this algorithm is the same as Eq. (14), except that $X = (\Delta t)^2 k_z^2 B_0^2 / S_A$. In the limit of $X \to 0$, the condition for stability is identical to the single-step case, i.e., Eq. (15). Again, a numerical solution of the dispersion relation for the split algorithm shows that numerical stability is achieved at all values $\omega_{ic} \tau_A$ if $C_A > |B_0|/\sqrt{2}$ and $C_H > \frac{1}{2}|B_0|/(\omega_{ci} \tau_A)$.

The same splitting procedure may be used with the Laplacian. However, again a first-order accurate method is obtained that is overly dispersive. If one wanted to use the Laplacian, the split method is better since the semi-implicit operator will not destroy the original MHD solution. Nevertheless, this is not recommended since the proper semi-implicit operator of Eq. (19) yields superior results.

3. Numerical Tests

In order to examine the effects of the different semi-implicit operators for the Hall term, we consider the case of an axisymmetric wave in a cylinder with a uniform axial magnetic field, $\mathbf{B} = B_0 \mathbf{z}$. Conducting wall boundaries are assumed at radius $r = a$. Density is assumed to be uniform and pressure is set to zero. The dispersion relation for the normal modes of this system [9] (switching to physical units) is

$$(\omega^2/\omega_{ci}^2) = (1 - \omega^2/k_z^2 V_A^2)\{1 - \omega^2/[V_A^2(k_z^2 + v_m^2)]\}, \tag{20}$$

where v_m corresponds to the zeros of the Bessel function, $J_1(v_m a) = 0$. When $(\omega/\omega_{ci}) \ll 1$ the usual fast and slow modes of MHD are recovered. We choose $(\omega_{ci} \tau_A) = 3.5$ in order to study the effect of the semi-implicit operators on the

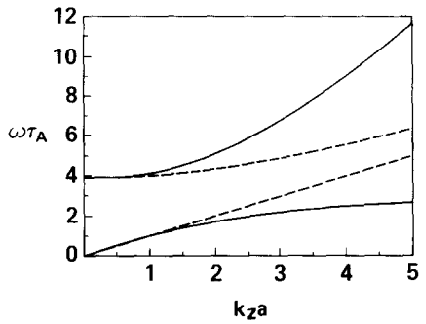

FIG. 1. Dispersion relation for the fast and slow modes with $\omega_{ci}\tau_A = 3.5$. The dashed line represents the modes without the Hall term.

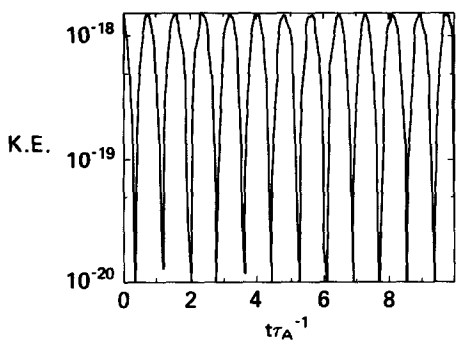

FIG. 2. Kinetic energy as a function of time for a fast mode with $k_z = 0.5$.

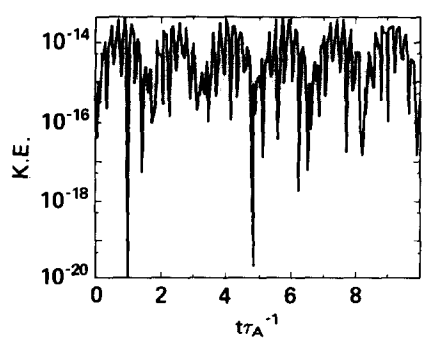

FIG. 3. Kinetic energy as a function of time for a fast mode with $k_z = 10$.

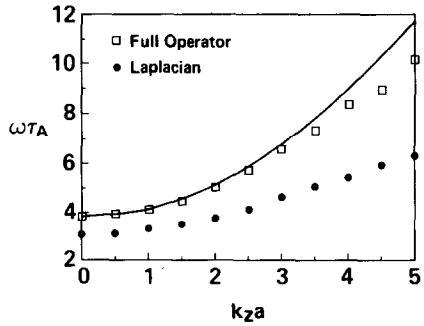

FIG. 4. Dispersion for the fast mode. The solid line shows the theoretical value and the points represent the results of single-step semi-implicit computations using the Laplacian and full operator.

solution. This value is chosen because it corresponds roughly to the size of the Hall term in ZT-40M reversed-field pinch experiments. Using this value, the dispersion relation for the fast and slow modes with the Hall term is plotted in Fig. 1. At large k_z the fast mode behaves like $\omega = (V_A^2/\omega_{ci})\, k_z^2$ and the slow mode asymptotes to $\omega = \omega_{ci}$.

Using the single step algorithms of Eqs. (8) and (9) and initializing the first cylindrical eigenmode ($m = 1$) with $v_1 a = 3.832$ (see Ref. [9]), the results of the semi-implicit method can be compared to the analytic dispersion relation. A time step of $\Delta t = 0.1\tau_A$ is used and the semi-implicit coefficients are set to $C_A = 0.55\, V_A$ and $C_H = 0.157\, V_A$. The time step was chosen to correspond to a typical value for 3-dimensional resistive MHD computations using the semi-implicit method. The computations were performed in a cylindrical code [5] with a spectral representation in the axial and azimuthal directions and second order finite differences in the radial direction. The maximum time step for an explicit leapfrog algorithm with the Hall term is $(\Delta t)^2 < 4/[k_z^2 a^2 (k_z^2 + k_r^2)\, V_A^2 (\omega_{ci}\tau_A)^{-2}]$, where $k_r = 2/\Delta r$, and Δr is the radial grid spacing. Figures 2 and 3 show the kinetic energy as a function of time for $k_z a = 0.5$ and $k_z a = 10$, respectively. Sixty-four radial gridpoints are used.

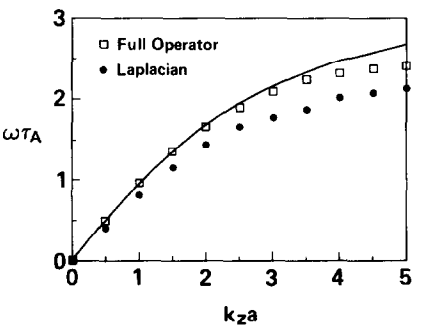

FIG. 5. Dispersion relation for the slow mode showing the theoretical value (solid line) and results of single-step computations.

The maximum explicit time steps for these two cases would be $\Delta t < 0.11\,\tau_A$ for $k_z a = 0.5$ and $\Delta t < 0.0055\,\tau_A$ for $k_z a = 10$. These conditions become much more severe when azimuthal variation is allowed. This maximum, $k_z a = 10$, is a typical value for the maximum $k_z a$ in a tokamak disruption calculation ($k_z a = 10$ corresponds to the $n = 20$ mode in an aspect ratio 2 tokamak). The frequency computed for the $k_z a = 0.5$ case is $\omega = 3.9\,\tau_A^{-1}$, which agrees with the theoretical value. The $k_z a = 10$ case for the fast mode is not resolved in time, since the theoretical frequency is $\omega = 120\,\tau_A^{-1}$. The rapid oscillation in Fig. 3 corresponds to the fast mode at a reduced frequency because of the large time step ($\Delta t = 0.1\,\tau_A$), while the slower oscillation is a result of the slow mode. The algorithm remains stable and there is no dissipation, as can be seen in Fig. 3. Figures 4 and 5 show the dispersion curves for the fast and slow modes. The solid line corresponds to the analytic result and the points represent the values obtained from the semi-implicit computations using the single-step algorithm for both the semi-implicit operator of Eq. (8) and the Laplacian operator. The algorithm using the fourth-order operator is accurate even at large $k_z a$. The Laplacian has poor accuracy at all $k_z a$. This poor accuracy is due both to the fact that the operator is only $O(\Delta t)$ accurate and because it acts isotropically, so that even when there is no variation along **B**, it affects the solution. The more accurate semi-implicit operator of Eq. (8) behaves properly, since its effect vanishes for $k_z = 0$.

The dispersion relation for the single step semi-implicit algorithms applied to the cylindrical problem is, in the limit of $\omega\,\Delta t \ll 2$,

$$\frac{\omega^2}{\omega_{ci}^2} = \frac{S_H}{k_z^2(k_z^2 + v_m^2)\,V_A^4}\left(\omega^2 - \frac{k_z^2 V_A^2}{S_A S_H}\right)\left(\omega^2 - \frac{(k_z^2 + v_m^2)\,V_A^2}{S_A S_H}\right), \tag{21}$$

with $S_A = 1 + (\Delta t)^2\,(k_z^2 + v_m^2)\,C_A^2$ and $S_H = 1 + (\Delta t)^2\,k_z^2(k_z^2 + v_m^2)\,C_H^2$ for the full operator and $S_H = 1 + \Delta t(k_z^2 + v_m^2)\,C_H$ for the Laplacian. These dispersion relations are plotted in Figs. 6 and 7 for the case of $\omega_{ci}\tau_A = 3.5$ and $\Delta t = 0.1\,\tau_A$. These curves

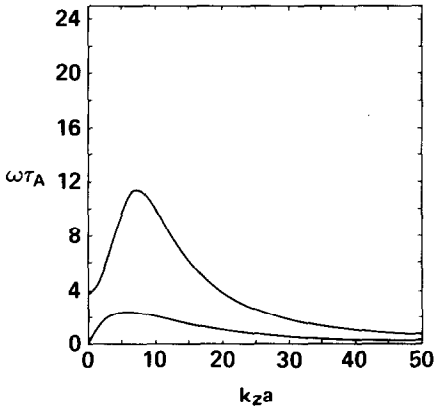

FIG. 6. Numerical dispersion relation for the single-step algorithm, Eq. (21), with the full operator, showing the fast (upper curve) and slow (lower curve) modes.

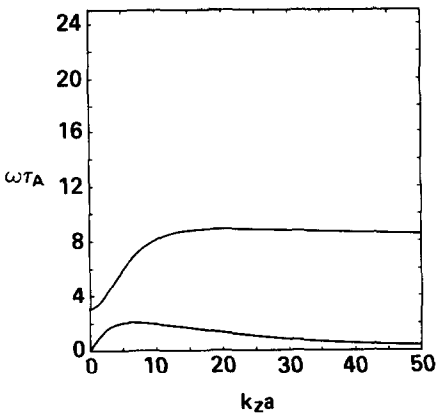

FIG. 7. Numerical dispersion relation for the single-step algorithm, Eq. (21), with the Laplacian operator, showing the fast (upper curve) and slow (lower curve) modes.

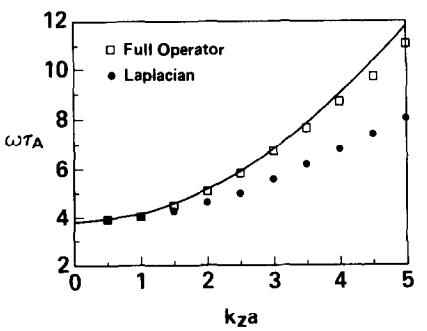

FIG. 8. Fast mode results, as in Fig. 4, using the split algorithm.

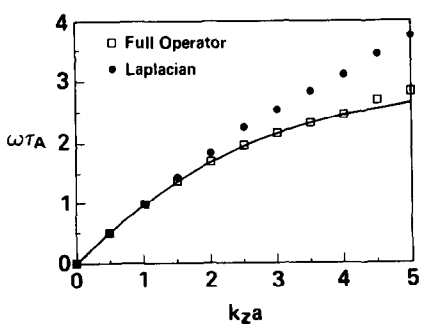

FIG. 9. Slow mode results, as in Fig. 5, using the split algorithm.

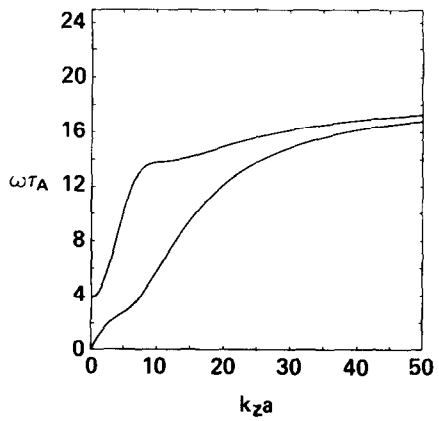

FIG. 10. Numerical dispersion relation for the split algorithm, Eq. (22), with the full operator, showing the fast (upper curve) and slow (lower curve) modes.

agree very well with the numerical results in Figs. 4 and 5 at low k_z. A negative aspect of these operators is that at very large k_z both drive the computed slow mode frequency toward zero. High k_z modes that are undamped will consequently appear spuriously at low frequencies. Additionally, the semi-implicit terms will cause these modes to have a negative group velocity ($\partial\omega/\partial k < 0$) at high k_z, leading to the propagation of information in the wrong direction. It is, therefore, necessary to include some dissipation at high k to prevent these errors from affecting the low frequency solution. This dissipation may be added as an artificial viscosity and is also useful to eliminate noise due to poorly resolved modes at high frequency.

The problem of negative group velocity may be eliminated by using the split algorithm [Eqs. (18) and (19)]. Figures 8 and 9 show the numerical and analytic

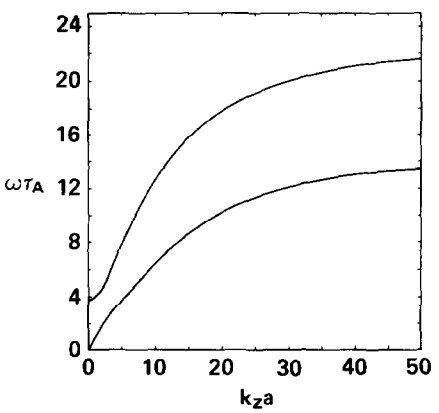

FIG. 11. Numerical dispersion relation for the split algorithm, Eq. (22), with the Laplacian operator, showing the fast (upper curve) and slow (lower curve) modes.

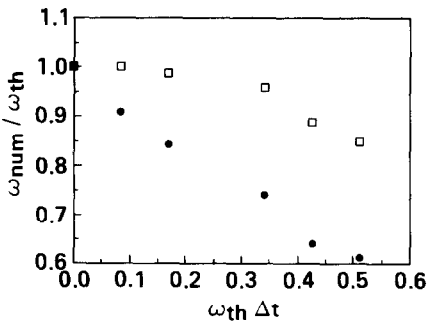

FIG. 12. Ratio of computed to theoretical frequency for the slow mode as a function of $\omega_{th}\,\Delta t$, using the single-step algorithm for the full operator (squares) and the Laplacian (solid circles).

results of using the split algorithm. Once again the full operator [Eq. (19)] is accurate even at large k_z. The Laplacian performs somewhat better than it does in the single-step method because when the Hall term vanishes (i.e., at low k_z) its effect disappears. The error for the Laplacian is nevertheless substantial whenever the Hall term is important. Unlike the single-step algorithm, the error now moves the dispersion relation in the direction of the ordinary MHD solution without the Hall term. The dispersion relation for this split algorithm is

$$\frac{\omega^2}{\omega_{ci}^2}=\frac{S_H}{k_z^2(k_z^2+\nu_m^2)\,V_A^4}\left(\omega^2-\frac{k_z^2V_A^2}{S_A}\right)\left(\omega^2-\frac{(k_z^2+\nu_m^2)\,V_A^2}{S_A}\right) \tag{22}$$

with S_A and S_H defined for the two operators as in Eq. (21). These dispersion relations are plotted in Figs. 10 and 11. Once again these curves agree with the above numerical results at low k_z. The dispersion for the split algorithm is superior to the single-step algorithm because the high k modes no longer have negative group velocity. The split algorithm, therefore, is the optimum choice using the operator of Eq. (19).

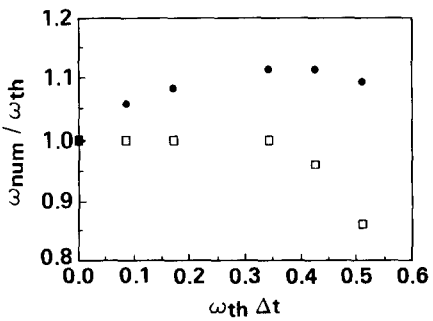

FIG. 13. Ratio of computed to theoretical frequency for the slow mode using the split algorithm for the full operator (squares) and the Laplacian (solid circles).

The effect of varying the time step size can be seen in Figs. 12 and 13. Initializing the same mode as above with $k_z a = 2$, the time step is varied from 0.001 to $0.6\,\tau_A$. The frequency for the slow mode in this case is $\omega\tau_A = 1.7$. Figures 12 and 13 show the ratio of the computed frequency to the theoretical frequency, $\omega_{\text{num}}/\omega_{\text{th}}$, as a function of $\omega_{\text{th}}\,\Delta t$ for the slow mode. The full operator is clearly superior to the Laplacian and, for this case, in which the Hall term is important but not dominant, the split algorithm is somewhat better than the single-step method. The split algorithm with the full operator is accurate within one percent out to $\omega_{\text{th}}\,\Delta t = 0.4$.

The semi-implicit operators for the Hall term and the Alfvén waves are both derived from equations with second derivatives with respect to time. The semi-implicit terms are then added to the corresponding first-order differential equations. For stability it is necessary to add the terms to one of the two first-order equations. In the semi-implicit MHD method only the momentum equation is modified and the magnetic field advance is still explicit. Then Hall terms are different in that they couple different components of the magnetic field, so it is not obvious to which single equation the semi-implicit terms should be applied. When the semi-implicit terms are applied to all components of the induction equation the algorithms described in Section 2 are obtained. The dispersion relation. Eq. (15), contains the denominator $1 + 2(\Delta t)^2 C_H k_z^4 + (\Delta t)^4 C_H^4 k_z^8$. All that is really necessary for stability is the term $2(\Delta t)^2 C_H^2 k_z^4$ in the denominator. This term would be a better operator at high k_z because it would eliminate the extraneous term $(\Delta t)^4 C_H^4 k_z^8$. Rather than applying the semi-implicit technique to one component, this less dispersive denominator may be obtained by using the semi-implicit advance only on every second time step. The semi-implicit coefficient, C_H, must be increased by a factor of 2 over the algorithm of Eqs. (8) and (9) for stability, but the resulting method is more efficient and less dispersive.

As an example of a practical application of this semi-implicit treatment, we consider an $m = 2$, $n = 1$ tokamak tearing mode. Toroidal effects are neglected, the aspect ratio is $R/a = 3$, $S = 10^4$, $B_z(r) = 1$ in normalized units, $\omega_{\text{ci}}\tau_A = 5$, and the q profile $q = 2\pi r B_z(r)/LB_\phi(r)$, is given by $q = [0.2(R/a)(1 - 0.5r^2/a^2)]^{-1}$. For simplicity, we have neglected the electron pressure gradient term which should normally be included in a tokamak computation. The inclusion of this term causes no additional numerical difficulties. This equilibrium is unstable to an $m = 2$, $n = 1$ resistive tearing mode with a growth rate of $\gamma = 4.3 \times 10^{-3}\,\tau_A^{-1}$. In order to test the different semi-implicit algorithms on this problem, we initialized this equilibrium in a spectral code retaining ten modes of a single helicity, $m = 1$, $n = 2$ to $m = 10$, $n = 20$. A random perturbation was then applied to this equilibrium and a time step of $\Delta t = 0.5\,\tau_A$ was used. One hundred radial grid points were used. Had this been an explicit calculation, the time step would have been limited by the Hall term to $\Delta t < 4.0 \times 10^{-4}\,\tau_A$. When this computation was performed with the single step algorithm using the Laplacian operator, the computed growth rate was much too high, $\gamma = 1.5 \times 10^{-2}\,\tau_A^{-1}$, due to a distortion of the time scales for resistive modes. As discussed previously with regard to linear waves, the split algorithm has no effect on the original MHD equations, but does introduce error into the Hall con-

tribution at large Δt. When the split algorithm is applied here with the Laplacian operator, using $\Delta t = 0.5\,\tau_A$, the growth rate is found to be $\gamma = 5.7 \times 10^{-3}\,\tau_A^{-1}$. This is identical to the growth rate for the MHD problem without the Hall terms present. The Laplacian operator had the effect of totally eliminating the Hall effect.

When the full operator is used in the split method, Eq. (18), the solution is accurate even at these large time steps. The coefficients of the semi-implicit operator are set so that at a given radius $C_{H\theta}$ and C_{Hz} are equal to 0.6 times the maximum values of B_θ and B_z, respectively, at that radius. The same computation as above was performed with this algorithm, again using a time step of $0.5\,\tau_A$. The growth rate was found to be $\gamma = 4.2 \times 10^{-3}\,\tau_A^{-1}$, giving good agreement with the true value. A key difference between this mode with the Hall term and the ideal mode is the presence of rotation. Figure 14 shows the saturated $m = 2$ island rotating with approximately the local diamagnetic drift frequency due to the Hall effect.

It should also be noted that this algorithm only addresses the most severe time step constraint imposed by the Hall term. Less severe constraints also arise from the $\mathbf{J} \cdot \nabla \mathbf{B}$ term in the induction equation. Therefore, once larger time steps are acceptable with the semi-implicit method, for some problems it may be necessary to consider techniques to eliminate these other constraints in order to perform the most efficient computations. The implementation of this algorithm, for either the full operator or the Laplacian, is not difficult in a spectral code having a Fourier

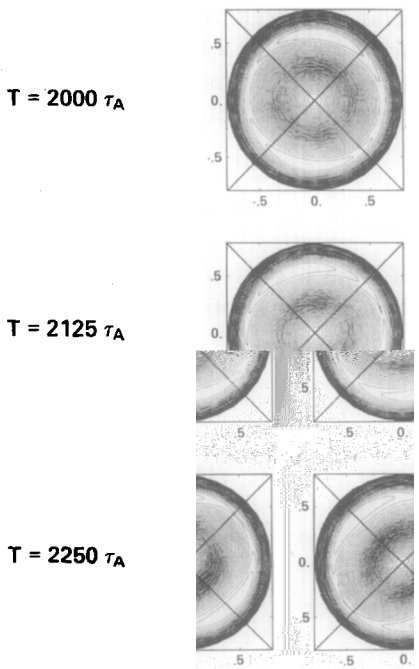

FIG. 14. Helical flux contours in a tokamak showing the Hall rotation of an $m = 2$, $n = 1$ island near saturation.

representation in two directions. In either case, the magnetic field advance requires the solution of a block tridiagonal system, which adds a negligible amount of additional computing time per time step when compared with an explicit advance. The solution becomes more complicated when two or more dimensions are in finite difference form. We have not addressed this situation here, but in such a case implementation of the full operator would be considerably more difficult than implementation of the Laplacian semi-implicit operator.

4. Conclusions

A semi-implicit technique for treating the Hall term has been developed. The semi-implicit operator is fourth order in $k_\parallel$ and second order in $k_\perp$. This operator eliminates the restrictive stability constraints on the time step due to the Hall term. It is possible to use a simpler operator, such as the Laplacian. The Laplacian, however, is very poor in terms of accuracy when large time steps are used. The most successful technique is found to be a split algorithm in which the new semi-implicit terms affect the Hall contribution. The method is simple to implement in spectral models and allows time steps to be comparable to those normally used for semi-implicit MHD computations. This semi-implicit technique should be appropriate for application to the simulation of reversed-field pinches, field-reversed configuration plasmas, and edge regions of tokamaks, for which the Hall term is essential. The method should also be useful for quasineutral hybrid computations since the time step constraint is the same as in the fluid model described here and only the magnetic field equation is involved.

Appendix

The fourth-order semi-implicit operator in Eq. (7) may be derived from Eq. (6) by the following procedure. Starting with Eq. (6),

$$\frac{\partial \mathbf{B}_1}{\partial t} = \frac{-1}{\omega_{ci}\tau_A} \nabla \times (\mathbf{J}_1 \times \mathbf{B}_0). \tag{A1}$$

Equation (A1) is differentiated with respect to time, giving

$$\frac{\partial^2 \mathbf{B}_1}{\partial t^2} = \frac{-1}{\omega_{ci}\tau_A} \nabla \times \left[\left(\nabla \times \frac{\partial \mathbf{B}_1}{\partial t} \right) \times \mathbf{B}_0 \right]. \tag{A2}$$

Equation (A1) is then substituted for the time derivative on the right side of Eq. (A2),

$$\frac{\partial^2 \mathbf{B}_1}{\partial t^2} = \frac{1}{(\omega_{ci}\tau_A)^2} \nabla \times \{ \nabla \times [\nabla \times (\mathbf{J}_1 \times \mathbf{B}_0)] \times \mathbf{B}_0 \}. \tag{A3}$$

Using $\nabla \cdot \mathbf{B} = 0$ and the uniformity of $\mathbf{B}_0$, this expression becomes

$$\frac{\partial^2 \mathbf{B}_1}{\partial t^2} = \frac{1}{(\omega_{ci}\tau_A)^2} \mathbf{B}_0 \cdot \nabla[\nabla \times (\mathbf{B}_0 \cdot \nabla \mathbf{J}_1)]. \tag{A4}$$

Equation (A4) can be converted to a simpler form by using the following relations derived from vector identities and the properties of $\mathbf{B}$ and $\mathbf{J}$:

$$\mathbf{B}_0 \cdot \nabla \mathbf{J}_1 = \nabla(\mathbf{J}_1 \cdot \mathbf{B}_0) + \mathbf{B}_0 \times \nabla^2 \mathbf{B}_1 \tag{A5}$$

and

$$\nabla \times (\mathbf{B}_0 \times \nabla^2 \mathbf{B}_1) = -(\mathbf{B}_0 \cdot \nabla) \nabla^2 \mathbf{B}_1. \tag{A6}$$

Applying these relations to Eq. (A4) gives

$$\frac{\partial^2 \mathbf{B}_1}{\partial t^2} = -\frac{1}{(\omega_{ci}\tau_A)^2} (\mathbf{B}_0 \cdot \nabla)^2 \nabla^2 \mathbf{B}_1, \tag{A7}$$

which is Eq. (7) of the text and defines the form of the fourth-order semi-implicit operator.

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